Supplementary appendix to Bugni et al. (2024)

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Abstract

This document provides results regarding CI^4_α that were omitted from the main text.

1 Proof of Theorems

Proof of part (c) of Theorem 1. We prove (3.3) by contradiction. That is, suppose that $\liminf_{N\to\infty}(P_N(\theta_N \in CI^4_\alpha) - P_N(\theta_N \in CI^j_\alpha)) < 0$ for some $j = 1, 2, 3$. We can then find a subsequence $\{k_N\}_{N\in\mathbb{N}}$ s.t.

$$\lim_{N\to\infty}(P_{k_N}(\theta_{k_N} \in CI^j_\alpha) - P_{k_N}(\theta_{k_N} \in CI^4_\alpha)) \; > \; 0 \quad \text{for some } j = 1, 2, 3. \tag{S-1}$$

The proof is completed by showing that (S-1) cannot hold.

By possibly taking a further subsequence,

$$\begin{aligned} &\left(\begin{array}{c} \theta_l(P_{k_N}), \theta_u(P_{k_N}), \sigma_l(P_{k_N}), \sigma_u(P_{k_N}), \rho(P_{k_N}), \sqrt{k_N}(\theta_u(P_{k_N}) - \theta_l(P_{k_N})), \\ \sqrt{k_N}(\theta_u(P_{k_N}) - \theta_l(P_{k_N}) - b_{k_N}), \sqrt{k_N}(\theta_l(P_{k_N}) - \theta_{k_N}), \sqrt{k_N}(\theta_{k_N} - \theta_u(P_{k_N})) \end{array} \right) \\ &\to \; (\theta_l, \theta_u, \sigma_l, \sigma_u, \rho, \mu, \eta, \Psi_l, \Psi_u). \end{aligned} \tag{S-2}$$

where $\{b_N\}_{N\geq 1}$ is the tuning parameter sequence used to implement CI^3_α.

We then divide the argument into six exhaustive cases, depending on possible values of (μ, Ψ_l, Ψ_u). In this regard, note that $\theta_N \in \Theta_I(P_N)^c$ implies that either (i) $\sqrt{k_N}(\theta_l(P_{k_N}) - \theta_{k_N}) > 0$ or (ii) $\sqrt{k_N}(\theta_{k_N} - \theta_u(P_{k_N})) > 0$. By taking limits, we conclude that either (i) $\Psi_l \geq 0$ or (ii) $\Psi_u \geq 0$. The result is completed by showing that none of the following exhaustive cases satisfy (S-1).

Case 1: $\mu = \infty$, $\Psi_l \geq 0$, and $\eta \in (0, \infty]$. Then, for any $j = 1, 2, 3$, consider the following derivation.

$$
\begin{aligned}
P_{k_N}(\theta_{k_N} \in CI_\alpha^4) &\overset{(1)}{\geq} P_{k_N}(\theta_{k_N} \in CI_\alpha^{4,a}) \\
&= P_{k_N}(\hat{\theta}_l - \hat{\sigma}_l c(\hat{\rho})/\sqrt{k_N} \leq \theta_{k_N} \leq \hat{\theta}_u + \hat{\sigma}_u c(\hat{\rho})/\sqrt{k_N}) \\
&= P_{k_N}\left(\left\{ \begin{array}{c} \{\sqrt{k_N}(\hat{\theta}_l - \theta_l(P_{k_N}))/\hat{\sigma}_l - c(\hat{\rho}) \leq -\sqrt{k_N}(\theta_l(P_{k_N}) - \theta_{k_N})/\hat{\sigma}_l\} \cap \\ -\sqrt{k_N}(\theta_l(P_{k_N}) - \theta_{k_N})/\hat{\sigma}_u - \sqrt{k_N}(\theta_u(P_{k_N}) - \theta_l(P_{k_N}))/\hat{\sigma}_u \leq \\ \sqrt{k_N}(\hat{\theta}_u - \theta_u(P_{k_N}))/\hat{\sigma}_u + c(\hat{\rho}) \end{array} \right\} \right) \\
&\overset{(2)}{\to} P(\{z_1 - c(\rho) \leq -\Psi_l/\sigma_l\} \cap \{-(\Psi_l + \mu)/\sigma_u \leq z_2\sqrt{1-\rho^2} + z_1\rho + c(\rho)\}) \\
&\overset{(3)}{=} P(\{z_1 - c(\rho) \leq -\Psi_l/\sigma_l\}) \\
&= \Phi(c(\rho) - \Psi_l/\sigma_l) \\
&\overset{(4)}{\geq} \Phi(\Phi^{-1}(1-\alpha) - \Psi_l/\sigma_l) \\
&\overset{(5)}{=} \lim P_{k_N}(\theta_{k_N} \in CI_\alpha^j),
\end{aligned} \tag{S-3}
$$

where (1) holds by $CI_\alpha^{4,a} \subseteq CI_\alpha^4$ with $CI_\alpha^{4,a} = [\hat{\theta}_l - \hat{\sigma}_l c(\hat{\rho})/\sqrt{k_N}, \hat{\theta}_u + \hat{\sigma}_u c(\hat{\rho})/\sqrt{k_N}]$ with $c(\cdot)$ as in Definition S-1, (2) by (S-2), Lemma S-1, and OBS (Definition 1), (3) by $\mu = \infty$, (4) by Lemma S-2, and (5) by part (a) of Lemma 1, $F_1(\infty, \sigma_l, \sigma_u) = \Phi^{-1}(1-\alpha)$, part (a) of Lemma 2, and part (c) of Lemma 3. Note that (S-3) implies that (S-1) fails.

Case 2: $\mu = \infty$, $\Psi_l \geq 0$, and $\eta \in [-\infty, 0]$. By this and Lemma 4, we deduce that $\rho = 1$ and $\sigma = \sigma_l = \sigma_u$. Then, for any $j = 1, 2, 3$, consider the following derivation.

$$
\begin{aligned}
P_{k_N}(\theta_{k_N} \in CI_\alpha^4) &\overset{(1)}{\geq} P_{k_N}(\theta_{k_N} \in CI_\alpha^{4,a}) \\
&= P_{k_N}\left(\left\{ \begin{array}{c} \{\sqrt{k_N}(\hat{\theta}_l - \theta_l(P_{k_N}))/\hat{\sigma}_l - c(\hat{\rho}) \leq -\sqrt{k_N}(\theta_l(P_{k_N}) - \theta_{k_N})/\hat{\sigma}_l\} \cap \\ -\sqrt{k_N}(\theta_l(P_{k_N}) - \theta_{k_N})/\hat{\sigma}_u - \sqrt{k_N}(\theta_u(P_{k_N}) - \theta_l(P_{k_N}))/\hat{\sigma}_u \\ \leq \sqrt{k_N}(\hat{\theta}_u - \theta_u(P_{k_N}))/\hat{\sigma}_u + c(\hat{\rho}) \end{array} \right\} \right) \\
&\overset{(2)}{\to} P(\{z_1 - c(1) \leq -\Psi_l/\sigma\}) \\
&\overset{(3)}{=} P(\{z_1 - \Phi^{-1}(1-\alpha/2) \leq -\Psi_l/\sigma\}) \\
&= \Phi(\Phi^{-1}(1-\alpha/2) - \Psi_l/\sigma) \\
&\overset{(4)}{\geq} \lim_{N\to\infty} P_{k_N}(\theta_{k_N} \in CI_\alpha^j),
\end{aligned} \tag{S-4}
$$

where (1) holds by $CI_\alpha^{4,a} \subseteq CI_\alpha^4$ with $CI_\alpha^{4,a} = [\hat{\theta}_l - \hat{\sigma}_l c(\hat{\rho})/\sqrt{k_N}, \hat{\theta}_u + \hat{\sigma}_u c(\hat{\rho})/\sqrt{k_N}]$ with $c(\cdot)$ as in Definition S-1, (2) by (S-2), Lemma S-1, and OBS (Definition 1), (3) by Lemma S-3, and (4) by

$\Phi^{-1}(1-\alpha/2) \geq \Phi^{-1}(1-\alpha)$, Lemma 1, $F_1(\infty,\sigma,\sigma) = \Phi^{-1}(1-\alpha)$, part (a) of Lemma 1, and parts (b) and (d) of Lemma 3. Note that (S-4) implies that (S-1) fails.

Case 3: $\mu < \infty$ and $\Psi_l \geq 0$. By $\mu \in \mathbb{R}_+$, it follows that $\theta_u(P_{k_N}) - \theta_l(P_{k_N}) \to 0$. By this and Lemma 4, we deduce that $\rho = 1$ and $\sigma = \sigma_l = \sigma_u$. Also, by repeating the argument in (A-5), we deduce that $\Phi^{-1}(1-\alpha/2) \geq G(\mu/\sigma)$. Then, for any $j = 1,2,3$, consider the following derivation.

$$\begin{aligned}
P_{k_N}(\theta_{k_N} \in CI^4_\alpha) &\overset{(1)}{\geq} P_{k_N}(\theta_{k_N} \in CI^{4,a}_\alpha) \\
&= P_{k_N}\left(\begin{array}{c} \{\sqrt{k_N}(\hat{\theta}_l - \theta_l(P_{k_N}))/\hat{\sigma}_l - c(\hat{\rho}) \leq -\sqrt{k_N}(\theta_l(P_{k_N}) - \theta_{k_N})/\hat{\sigma}_l\} \cap \\ \left\{ \begin{array}{c} -\sqrt{k_N}(\theta_l(P_{k_N}) - \theta_{k_N})/\hat{\sigma}_u - \sqrt{k_N}(\theta_u(P_{k_N}) - \theta_l(P_{k_N}))/\hat{\sigma}_u \\ \leq \sqrt{k_N}(\hat{\theta}_u - \theta_u(P_{k_N}))/\hat{\sigma}_u + c(\hat{\rho}) \end{array} \right\} \end{array} \right) \\
&\overset{(2)}{\to} P(\{z_1 - c(1) \leq -\Psi_l/\sigma\} \cap \{-(\Psi_l + \mu)/\sigma \leq z_1 + c(1)\}) \\
&\overset{(3)}{=} \Phi((\Psi_l + \mu)/\sigma + \Phi^{-1}(1-\alpha/2)) - \Phi(\Psi_l/\sigma - \Phi^{-1}(1-\alpha/2)) \\
&\overset{(4)}{\geq} \lim_{N\to\infty} P_{k_N}(\theta_{k_N} \in CI^j_\alpha),
\end{aligned} \tag{S-5}$$

where (1) holds by $CI^{4,a}_\alpha \subseteq CI^4_\alpha$ with $CI^{4,a}_\alpha = [\hat{\theta}_l - \hat{\sigma}_l c(\hat{\rho})/\sqrt{k_N}, \hat{\theta}_u + \hat{\sigma}_u c(\hat{\rho})/\sqrt{k_N}]$ with $c(\cdot)$ as in Definition S-1, (2) by (S-2), Lemma S-1, and OBS (Definition 1), (3) by Lemma S-3, and (4) by $\Phi^{-1}(1-\alpha/2) \geq G(\mu/\sigma)$, Lemma 1, $F_1(\mu,\sigma,\sigma) = G(\mu/\sigma)$, part (a) of Lemma 2, and part (e) of Lemma 3. Note that (S-5) implies that (S-1) fails.

Case 4: $\mu = \infty$, $\Psi_u \geq 0$, and $\eta \in (0,\infty]$. This case is analogous to case 1 and, therefore, omitted.

Case 5: $\mu = \infty$, $\Psi_u \geq 0$, and $\eta \in [-\infty,0]$. This case is analogous to case 2 and, therefore, omitted.

Case 6: $\mu < \infty$ and $\Psi_u \geq 0$. This case is analogous to case 3 and, therefore, omitted.

To conclude the proof, it suffices to verify (3.3) holds strictly for suitably chosen sequences $\{(P_N,\theta_N) \in \mathcal{P} \times \Theta_I(P_N)^c\}_{N\in\mathbb{N}}$. We use Example 1 with $\underline{Y} = 0$, $\overline{Y} = 1$, $\{Y_i|Z_i = 1\} \sim Be(1/2)$, and $Z_i \sim Be(1-\pi(P_N))$, where $\pi(P_N) = (\ln N)/\sqrt{N} + a_1/\sqrt{N} \downarrow 0$ for some $a_1 > 0$. We consider coverage of $\theta_N = \theta_l(P_N) - a_2/\sqrt{N} \in \Theta_I(P_N)^c$ for some $a_2 > 0$ and CI^3_α implemented with $b_N = (\ln N)/\sqrt{N}$ and the full sample the data (i.e., $\lambda = 1$).

By repeating arguments presented in the proof of Theorem 2, we deduce that $(\theta_l,\theta_u,\sigma_l,\sigma_u,\rho,\mu,\eta,\Psi_l) = (1/2,1/2,1/2,1/2,1,\infty,a_1,a_2)$, and part (c) of Lemma 3 implies that

$$\lim_{N\to\infty} P_N(\theta_N \in CI^3_\alpha) = \Phi(\Phi^{-1}(1-\alpha) - 2\sqrt{\lambda}a_2). \tag{S-6}$$

In turn, Lemmas 1, 2, S-3, and S-4 imply that

$$\begin{aligned}
\lim_{N\to\infty} P_N(\theta_N \in CI^j_\alpha) &= \Phi(\Phi^{-1}(1-\alpha) - 2\sqrt{\lambda}a_2) \qquad \text{for } j = 1,2, \\
\lim_{N\to\infty} P_N(\theta_N \in CI^4_\alpha) &= \Phi(\Phi^{-1}(1-\alpha/2) - 2\sqrt{\lambda}a_2).
\end{aligned} \tag{S-7}$$

The desired result follows from (S-6), (S-7), and $\Phi^{-1}(1-\alpha) < \Phi^{-1}(1-\alpha/2)$. For instance, using $a_1 = 1.5$, $a_2 = 0.01$, and $\alpha = 0.05$ yields $\lim_{N\to\infty} P_N(\theta_N \in CI^4_\alpha) = 0.974 > 0.948 \lim_{N\to\infty} P_N(\theta_N \in CI^1_\alpha) = \lim_{N\to\infty} P_N(\theta_N \in CI^2_\alpha) = \lim_{N\to\infty} P_N(\theta_N \in CI^3_\alpha)$. ■

Proof of Theorem 5. To show this result, we construct specific sequences where (3.9) and (3.10) can occur. We focus on sequences $\{(P_N, \theta_N) \in \mathcal{P} \times \Theta_I(P_N)^c\}_{N\in\mathbb{N}}$ s.t.

$$\begin{pmatrix} \theta_l(P_N), \theta_u(P_N), \sigma^E_l(P_N), \sigma^E_u(P_N), \rho^E(P_N), \sigma^I_l(P_N), \sigma^I_u(P_N), \rho^I(P_N) \\ \sqrt{N}(\theta_u(P_N) - \theta_l(P_N)), \sqrt{N}(\theta_l(P_N) - \theta_N), \sqrt{N}(\theta_N - \theta_u(P_N)) \end{pmatrix}$$
$$\to (\theta_l, \theta_u, \sigma^E_l, \sigma^E_u, \rho^E, \sigma^I_l, \sigma^I_u, \rho^I, \mu, \Psi_l, \Psi_u). \tag{S-8}$$

with $\Psi_l \geq 0$ and $\mu \in \mathbb{R}_+$. By Lemma 4, $\rho^E = \rho^I = 1$ and $\sigma^E_l = \sigma^E_u$, and $\sigma^I_l = \sigma^I_u$.

We can construct concrete sequences in the context of Example 1. In particular, we use Example 1 with $\underline{Y} = 0$, $\overline{Y} = 1$, $\{Y_i|Z_i = 1\} \sim Be(1/2)$, and $Z_i \sim Be(1-\pi(P_N))$, where $\pi(P_N) = a_1/\sqrt{N} \downarrow 0$ for some $a_1 > 0$. We consider coverage of $\theta_N = \theta_l(P_N) - a_2/\sqrt{N} \in \Theta_I(P_N)^c$ for some $a_2 > 0$ and CI^4_α implemented with the subsample of the data $\{(Y_i, Z_i)\}_{i=1}^{\lfloor \lambda N \rfloor}$ for $\lambda \in (0, 1]$. In this context, we consider two estimators for the bounds. The efficient estimator uses the full sample, while the inefficient estimator uses only a fraction $\lambda = a_3 \in (0, 1)$ of the sample. By repeating arguments in Theorem 2, we deduce that (S-8) holds with

$$(\theta_l, \theta_u, \sigma^E_l, \sigma^E_u, \rho^E, \sigma^I_l, \sigma^I_u, \rho^I, \mu, \Psi_l) \;=\; \Big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2\sqrt{a_3}}, \frac{1}{2\sqrt{a_3}}, 1, a_1, a_2\Big).$$

Part (b) of Lemma S-4 then yields

$$\lim_{N\to\infty} P_N(\theta_N \in CI^{4,E}_\alpha) \;=\; \Phi\big(2(a_2 + a_1) + \Phi^{-1}(1-\alpha/2)\big) - \Phi\big(2a_2 - \Phi^{-1}(1-\alpha/2)\big),$$
$$\lim_{N\to\infty} P_N(\theta_N \in CI^{4,I}_\alpha) \;=\; \Phi\big(2\sqrt{a_3}(a_2 + a_1) + \Phi^{-1}(1-\alpha/2)\big) - \Phi\big(2\sqrt{a_3}a_2 - \Phi^{-1}(1-\alpha/2)\big).$$

We now verify the strict inequalities. To obtain (3.9), set $a_1 = a_2 = 1$, $a_3 = 0.3$, and $\alpha = 0.05$, which give

$$\lim_{N\to\infty} P_N(\theta_N \in CI^{4,E}_\alpha) \;=\; 0.48 \;<\; 0.81 \;=\; \lim_{N\to\infty} P_N(\theta_N \in CI^{4,I}_\alpha).$$

To obtain (3.10), set $a_1 = 0.15$, $a_2 = 0.01$, $a_3 = 0.3$, and $\alpha = 0.05$, which give

$$\lim_{N\to\infty} P_N(\theta_N \in CI^{4,E}_\alpha) \;=\; 0.963 \;>\; 0.958 \;=\; \lim_{N\to\infty} P_N(\theta_N \in CI^{4,I}_\alpha).$$

This completes the proof. ■

2 Auxiliary material

This section collects auxiliary definitions and intermediate results that are useful to prove parts (e)-(f) of Theorem 1 and Theorem 5.

Definition S-1. *For any $\rho \in [-1,1]$, $c(\rho)$ is the unique c that solves*

$$\inf_{\Delta \geq 0} P\left(\begin{array}{c} \{z_1 - \Delta - c \leq 0 \leq \rho z_1 + z_2\sqrt{1-\rho^2} + c\} \cup \\ \{|(1+\rho)z_1 + z_2\sqrt{1-\rho^2} - \Delta| \leq \sqrt{2+2\rho}\Phi^{-1}(1-\alpha/2)\} \end{array} \right) \;=\; 1-\alpha,$$

where $z = (z_1, z_2) \sim \mathcal{N}(\mathbf{0}_{2\times 1}, \mathbf{I}_{2\times 2})$.

Remark S-1. *By definition, Stoye (2020) uses $c^4 = c(\hat{\rho})$. To see this, observe that $z = (z_1, z_2) \sim \mathcal{N}(\mathbf{0}_{2\times 1}, \mathbf{I}_{2\times 2})$ is equivalent to $\tilde{z}(\rho) = (\tilde{z}_1(\rho), \tilde{z}_2(\rho)) := (z_1, \rho z_1 + z_2\sqrt{1-\rho^2}) \sim \mathcal{N}(\mathbf{0}_{2\times 1}, [1,\rho;\rho,1])$. Therefore, $c(\rho)$ can equivalently be characterized as the unique c solving*

$$\inf_{\Delta \geq 0} P\left(\begin{array}{c} \{\tilde{z}_1(\rho) - \Delta - c \leq 0 \leq \tilde{z}_2(\rho) + c\} \cup \\ \{|\tilde{z}_1(\rho) + \tilde{z}_2(\rho) - \Delta| \leq \sqrt{2+2\rho}\,\Phi^{-1}(1-\alpha/2)\} \end{array} \right) \;=\; 1-\alpha.$$

If we set $\rho = \hat{\rho}$ and condition on $\hat{\rho}$, the above expression coincides with the one in Stoye (2020).

Lemma S-1. *$c(\rho)$ in Definition S-1 is a continuous function of $\rho \in [-1,1]$.*

Proof. Define the function $G : [-1,1] \times \mathbb{R}_+ \times \mathbb{R}_+$ as follows:

$$G(\rho, \Delta, c) \;=\; P\left(\begin{array}{c} \{z_1 - \Delta - c \leq 0 \leq \rho z_1 + z_2\sqrt{1-\rho^2} + c\} \cup \\ \{|(1+\rho)z_1 + z_2\sqrt{1-\rho^2} - \Delta| \leq \sqrt{2+2\rho}\Phi^{-1}(1-\alpha/2)\} \end{array} \right).$$

It is easy to see that G is continuous in all its arguments. By definition,

$$c(\rho) \;=\; \left\{ c : \inf_{\Delta \geq 0} G(\rho, \Delta, c) = 1-\alpha \right\},$$

where the uniqueness of the solution follows from Stoye (2020).

Suppose this is not the case, i.e., $\{\rho_m\}_{m\in\mathbb{N}}$ with $\rho_m \to \rho \in [-1,1]$ and $c(\rho_m) \not\to c(\rho)$. By possibly taking a subsequence, we have that $c(\rho_m) \to \bar{c} \neq c(\rho)$ as $m \to \infty$, where $\bar{c} \in [0,\infty]$. By definition of $c(\rho_m)$,

$$\inf_{\Delta \geq 0} G(\rho_m, \Delta, c(\rho_m)) \;=\; 1-\alpha. \tag{S-9}$$

Since $c(\rho)$ is unique and $\bar{c} \neq c(\rho)$, we have $\inf_{\Delta\geq 0} G(\rho, \Delta, \bar{c}) \neq 1-\alpha$. There are two cases: $\inf_{\Delta\geq 0} G(\rho, \Delta, \bar{c}) > 1-\alpha$ or $\inf_{\Delta\geq 0} G(\rho, \Delta, \bar{c}) < 1-\alpha$. To complete the proof, it suffices to show that both cases are contradictory.

Case 1: $\inf_{\Delta\geq 0} G(\rho, \Delta, \bar{c}) > 1-\alpha$. Then, $\exists \varepsilon > 0$ s.t. $\inf_{\Delta\geq 0} G(\rho, \Delta, \bar{c}) \geq 1-\alpha+\varepsilon$. Then, for all $\Delta \geq 0$, $G(\rho, \Delta, \bar{c}) \geq 1-\alpha+\varepsilon$. Fix $\Delta \geq 0$ arbitrarily. By continuity of G, $\lim_{m\to\infty} G(\rho_m, \Delta, c(\rho_m)) =$

$G(\rho, \Delta, \bar{c}) \geq 1 - \alpha + \varepsilon$. Therefore, $\exists M$ s.t. for all $m \geq M$, $G(\rho_m, \Delta, c(\rho_m)) \geq 1 - \alpha + \varepsilon/2$. Since $\Delta \geq 0$ was arbitrary, this implies

$$\inf_{\Delta \geq 0} G(\rho_m, \Delta, c(\rho_m)) \;\geq\; 1 - \alpha + \varepsilon/2 \;>\; 1 - \alpha,$$

which is a contradiction to (S-9).

Case 2: $\inf_{\Delta \geq 0} G(\rho, \Delta, \bar{c}) < 1 - \alpha$. Then, $\exists \varepsilon > 0$ s.t. $\inf_{\Delta \geq 0} G(\rho, \Delta, \bar{c}) \leq 1 - \alpha - \varepsilon$. Then, $\exists \bar{\Delta} \geq 0$ s.t. $G(\rho, \bar{\Delta}, \bar{c}) \leq 1 - \alpha - \varepsilon/2$. By continuity of G, $\lim_{m \to \infty} G(\rho_m, \bar{\Delta}, c(\rho_m)) = G(\rho, \bar{\Delta}, \bar{c}) \leq 1 - \alpha - \varepsilon/2$. Therefore, $\exists M$ s.t. for all $m \geq M$, $G(\rho_m, \bar{\Delta}, c(\rho_m)) \leq 1 - \alpha - \varepsilon/4$. Then, for all $m \geq M$,

$$\inf_{\Delta \geq 0} G(\rho_m, \Delta, c(\rho_m)) \;\leq\; G(\rho_m, \bar{\Delta}, c(\rho_m)) \;\leq\; 1 - \alpha - \varepsilon/4 \;<\; 1 - \alpha,$$

which is a contradiction to (S-9). ■

Lemma S-2. *For any $\rho \in [-1, 1]$ and $\alpha \in (0, 1)$, $c(\rho)$ in Definition S-1 satisfies $c(\rho) \geq \Phi^{-1}(1-\alpha)$.*

Proof. Fix $\rho \in [-1, 1]$ arbitrarily. Assume the result is false, i.e., $c(\rho) < \Phi^{-1}(1 - \alpha)$. Define

$$\Pi(c) \;=\; \inf_{\Delta \geq 0} P\left(\begin{array}{c} \{z_1 - \Delta - c \leq 0 \leq \rho z_1 + z_2\sqrt{1-\rho^2} + c\} \cup \\ \left\{ \begin{array}{c} \Delta - \sqrt{2+2\rho}\,\Phi^{-1}(1-\alpha/2) \leq \\ z_1(1+\rho) + z_2\sqrt{1-\rho^2} \leq \Delta + \sqrt{2+2\rho}\,\Phi^{-1}(1-\alpha/2) \end{array} \right\} \end{array} \right).$$

First, we show that $\Pi(c)$ is strictly increasing in c. We have two cases, depending on whether a minimizer of the right-hand side, denoted by Δ^*, is finite. If $\Delta^* < \infty$,

$$\Pi(c) \;=\; P\left(\begin{array}{c} \{z_1 - \Delta^* - c \leq 0 \leq \rho z_1 + z_2\sqrt{1-\rho^2} + c\} \cup \\ \left\{ \begin{array}{c} \Delta^* - \sqrt{2+2\rho}\,\Phi^{-1}(1-\alpha/2) \leq \\ z_1(1+\rho) + z_2\sqrt{1-\rho^2} \leq \Delta^* + \sqrt{2+2\rho}\,\Phi^{-1}(1-\alpha/2) \end{array} \right\} \end{array} \right),$$

and the right-hand side is strictly increasing in c. If $\Delta^* = \infty$,

$$\Pi(c) \;=\; P\big(0 \leq \rho z_1 + z_2\sqrt{1-\rho^2} + c\big),$$

which is also strictly increasing in c.

By $\Pi(c(\rho)) = 1 - \alpha$ and $c(\rho) < \Phi^{-1}(1 - \alpha)$, the strict increasingness of $\Pi(c)$ implies that $\Pi(\Phi^{-1}(1 - \alpha)) > 1 - \alpha$. Then, $\exists \varepsilon > 0$ s.t. $\Pi(\Phi^{-1}(1 - \alpha)) \geq 1 - \alpha + \varepsilon$. Then, for any $\Delta \geq 0$,

$$P\left(\begin{array}{c} \{z_1 - \Delta - \Phi^{-1}(1-\alpha) \leq 0 \leq \rho z_1 + z_2\sqrt{1-\rho^2} + \Phi^{-1}(1-\alpha)\} \cup \\ \{\Delta - \sqrt{2+2\rho}\,\Phi^{-1}(1-\alpha/2) \leq z_1(1+\rho) + z_2\sqrt{1-\rho^2} \leq \Delta + \sqrt{2+2\rho}\,\Phi^{-1}(1-\alpha/2)\} \end{array} \right)$$
$$\geq\; 1 - \alpha + \varepsilon/2.$$

Now evaluate this inequality for a sequence $\Delta_m \to \infty$ to obtain

$$P(0 \le \rho z_1 + z_2\sqrt{1-\rho^2} + \Phi^{-1}(1-\alpha)) \ \ge \ 1-\alpha+\varepsilon/2.$$

Using $\tilde{z} = \rho z_1 + z_2\sqrt{1-\rho^2} \sim N(0,1)$, we obtain

$$1-\alpha+\varepsilon/2 \ \le \ P(0 \le \tilde{z} + \Phi^{-1}(1-\alpha)) \ = \ P(-\Phi^{-1}(1-\alpha) \le \tilde{z}) \ = \ 1-\alpha,$$

which is a contradiction. ■

Lemma S-3. *For any $\alpha \in (0,1)$, $c(\rho)$ in Definition S-1 satisfies $c(1) = \Phi^{-1}(1-\alpha/2)$.*

Proof. By definition,

$$c(1) \ = \ \left\{ c : \inf_{\Delta \ge 0} P\left(\begin{array}{c} \{-c \le z \le \Delta + c\} \quad \cup \\ \{\Delta/2 - \Phi^{-1}(1-\alpha/2) \le z \le \Phi^{-1}(1-\alpha/2) + \Delta/2\} \end{array} \right) = 1-\alpha \right\}.$$

Given any $c \in \mathbb{R}$, consider the minimization of

$$G(\Delta) = \ \Phi(\max\{\Phi^{-1}(1-\alpha/2) + \Delta/2, \Delta + c\}) - \Phi(\min\{\Delta/2 - \Phi^{-1}(1-\alpha/2), -c\})$$

with respect to $\Delta \ge 0$. We have four cases.

Case 1: $\Delta \ge 2\Phi^{-1}(1-\alpha/2) - 2c \ge 0$. Then, $\Delta + c \ge \Phi^{-1}(1-\alpha/2) + \Delta/2$ and $-c \le \Delta/2 - \Phi^{-1}(1-\alpha/2)$, and so $G(\Delta) = \Phi(\Delta + c) - \Phi(-c)$. The function is increasing in Δ, so it is minimized by setting $\Delta = 2\Phi^{-1}(1-\alpha/2) - 2c$.

Case 2: $\Delta \ge 2\Phi^{-1}(1-\alpha/2) - 2c$ and $2\Phi^{-1}(1-\alpha/2) - 2c < 0$. The same argument with the constraint $\Delta \ge 0$ yields $\Delta = 0$.

Case 3: $\Delta < 2\Phi^{-1}(1-\alpha/2) - 2c$ and $2\Phi^{-1}(1-\alpha/2) - 2c \ge 0$. Then, $G(\Delta) = \Phi(\Phi^{-1}(1-\alpha/2) + \Delta/2) - \Phi(\Delta/2 - \Phi^{-1}(1-\alpha/2))$. The function is decreasing in Δ, so it is minimized by setting $\Delta = 2\Phi^{-1}(1-\alpha/2) - 2c$.

Case 4: $\Delta < 2\Phi^{-1}(1-\alpha/2) - 2c$ and $2\Phi^{-1}(1-\alpha/2) - 2c < 0$. The same argument with the constraint $\Delta \ge 0$ yields $\Delta = 0$.

Combining all cases,

$$\inf_{\Delta \ge 0} G(\Delta) \ = \ \left\{ \begin{array}{c} I\{\Phi^{-1}(1-\alpha/2) \ge c\}(\Phi(2\Phi^{-1}(1-\alpha/2) - c) - \Phi(-c)) \\ +I\{\Phi^{-1}(1-\alpha/2) < c\}(\Phi(c) - \Phi(-c)) \end{array} \right\}. \tag{S-10}$$

Now, we solve for c s.t. (S-10) equals $1-\alpha$ which, by definition, equals $c(1)$.

We now show that $\Phi^{-1}(1-\alpha/2) \ge c$. Suppose otherwise that $\Phi^{-1}(1-\alpha/2) < c$. Then, the right-hand side of (S-10) equals $\Phi(c) - \Phi(-c)$. Then, the desired solution c satisfies $\Phi(c) - \Phi(-c) = 1-\alpha$, which yields $\Phi(c) = 1 - \alpha/2$, and, so, $c = \Phi^{-1}(1 - \alpha/2)$, which contradicts the premise that $\Phi^{-1}(1 - \alpha/2) < c$.

Since $\Phi^{-1}(1 - \alpha/2) \geq c$, the right-hand side of (S-10) equals $\Phi(2\Phi^{-1}(1 - \alpha/2) - c) - \Phi(-c)$. Then, the desired solution c satisfies

$$\Phi(2\Phi^{-1}(1 - \alpha/2) - c) - \Phi(-c) \; = \; 1 - \alpha. \tag{S-11}$$

Note that the left-hand side of (S-11) is strictly increasing in c. Therefore, the unique solution is $c = \Phi^{-1}(1 - \alpha/2)$, as desired. ■

Lemma S-4. *Let $\alpha \in (0, 0.5)$ and Assumption 1 hold. Let $\{(P_N, \theta_N) \in \mathcal{P} \times \Theta_I(P_N)^c\}_{N \in \mathbb{N}}$ be a sequence s.t.*

$$\begin{aligned} &\left(\begin{array}{c} \theta_l(P_N), \theta_u(P_N), \sigma_l(P_N), \sigma_u(P_N), \rho(P_N), \\ \sqrt{N}(\theta_u(P_N) - \theta_l(P_N)), \sqrt{N}(\theta_l(P_N) - \theta_N), \sqrt{N}(\theta_N - \theta_u(P_N)) \end{array} \right) \\ &\to \; (\theta_l, \theta_u, \sigma_l, \sigma_u, \rho, \mu, \Psi_l, \Psi_u). \end{aligned} \tag{S-12}$$

Moreover, assume that $\Psi_l \geq 0$. Then,

(a) If $\mu = \infty$, $P_N(\theta_N \in CI_\alpha^4) \to \Phi(c(\rho) - \Psi_l/\sigma_l)$.

(b) If $\mu \in \mathbb{R}_+$, then $\rho = 1$, $\sigma_l = \sigma_u$, and, if we denote $\sigma = \sigma_l = \sigma_u$, we get

$$P_N(\theta_N \in CI_\alpha^4) \; \to \; \Phi((\Psi_l + \mu)/\sigma + \Phi^{-1}(1 - \alpha/2)) - \Phi(\Psi_l/\sigma - \Phi^{-1}(1 - \alpha/2)).$$

Proof. As a preliminary result, note that

$$\Psi_u \; = \; -\lim(\sqrt{N}(\theta_u(P_N) - \theta_l(P_N)) + \sqrt{N}(\theta_l(P_N) - \theta_N)) \; = \; -\mu - \Psi_l. \tag{S-13}$$

We divide the proof into two parts.

Part (a): In this case, $\mu = \infty$. Then $\Psi_l \geq 0$ and, by (S-13), $\Psi_u = -\infty$. Hence,

$$\begin{aligned} &P_N(\theta_N \in CI_\alpha^4) \\ &= \; P_N(\theta_N \in CI_\alpha^{4,a} \cup CI_\alpha^{4,b}) \\ &= \; P_N \left(\begin{array}{c} \left\{ \begin{array}{c} \{\sqrt{N}(\hat{\theta}_l - \theta_l(P_N))/\hat{\sigma}_l - c(\hat{\rho}) \leq -\sqrt{N}(\theta_l(P_N) - \theta_N)/\hat{\sigma}_l\} \cap \\ \left\{ \begin{array}{c} -\sqrt{N}(\theta_l(P_N) - \theta_N)/\hat{\sigma}_u - \sqrt{N}(\theta_u(P_N) - \theta_l(P_N))/\hat{\sigma}_u \\ \leq \sqrt{N}(\hat{\theta}_u - \theta_u(P_N))/\hat{\sigma}_u + c(\hat{\rho}) \end{array} \right\} \end{array} \right\} \\ \cup \left\{ \begin{array}{c} \sqrt{N}(\theta_N - \theta_u(P_N))/\hat{\sigma}_u - \sqrt{N}(\theta_l(P_N) - \theta_N)/\hat{\sigma}_l - \sqrt{2 + 2\hat{\rho}}\Phi^{-1}(1 - \alpha/2) \\ \leq \sqrt{N}(\hat{\theta}_l - \theta_l(P_N))/\hat{\sigma}_l + \sqrt{N}(\hat{\theta}_u - \theta_u(P_N))/\hat{\sigma}_u \leq \\ \sqrt{N}(\theta_N - \theta_u(P_N))/\hat{\sigma}_u - \sqrt{N}(\theta_l(P_N) - \theta_N)/\hat{\sigma}_l + \sqrt{2 + 2\hat{\rho}}\Phi^{-1}(1 - \alpha/2) \end{array} \right\} \end{array} \right) \end{aligned}$$

$$\stackrel{(1)}{\to} P\left(\begin{array}{c}\{\{z_1 - c(\rho) \leq -\Psi_l/\sigma_l\} \cap \{-\Psi_l/\sigma_l - \mu/\sigma_u \leq \rho z_1 + z_2\sqrt{1-\rho^2} + c(\rho)\}\}\cup \\ \left\{\begin{array}{c}\Psi_u/\sigma_u - \Psi_l/\sigma_l - \sqrt{2+2\rho}\Phi^{-1}(1-\alpha/2) \leq (1+\rho)z_1 + z_2\sqrt{1-\rho^2} \\ \leq \Psi_u/\sigma_u - \Psi_l/\sigma_l + \sqrt{2+2\rho}\Phi^{-1}(1-\alpha/2)\end{array}\right\}\end{array}\right)$$

$$\stackrel{(2)}{=} P(z_1 - c(\rho) \leq -\Psi_l/\sigma_l) \;=\; \Phi(c(\rho) - \Psi_l/\sigma_l),$$

as desired, where (1) holds by (S-12), Lemma S-1, and OBS (Definition 1), and (2) by $\mu = \infty$, $\Psi_l \geq 0$, and (S-13), which implies that $\Psi_u = -\infty$.

Part (b): In this case, $\mu \in \mathbb{R}_+$. Lemma 4 then implies that $\rho = 1$ and $\sigma_l = \sigma_u$. Let $\sigma = \sigma_l = \sigma_u$. By a similar derivation as in part (a),

$$\begin{aligned} P_N(\theta_N \in CI_\alpha^4) &\stackrel{(1)}{\to} P\left(\begin{array}{c}\{\{z_1 - c(1) \leq -\Psi_l/\sigma\} \cap \{-(\Psi_l+\mu)/\sigma \leq z_1 + c(1)\}\} \\ \cup\left\{\begin{array}{c}(\Psi_u - \Psi_l)/\sigma - 2\Phi^{-1}(1-\alpha/2) \leq 2z_1 \\ \leq (\Psi_u - \Psi_l)/\sigma + 2\Phi^{-1}(1-\alpha/2)\end{array}\right\}\end{array}\right) \\ &\stackrel{(2)}{=} P(-(\Psi_l+\mu)/\sigma - \Phi^{-1}(1-\alpha/2) \leq z_1 \leq -\Psi_l/\sigma + \Phi^{-1}(1-\alpha/2)) \\ &= \Phi((\Psi_l+\mu)/\sigma + \Phi^{-1}(1-\alpha/2)) - \Phi(\Psi_l/\sigma - \Phi^{-1}(1-\alpha/2)),\end{aligned}$$

where (1) holds by (S-12), Lemma S-1, and OBS (Definition 1), and (2) by Lemma S-3, $\mu \geq 0$, $\Psi_l \geq 0$, and (S-13), which implies that $\Psi_u = -\mu - \Psi_l$. ■

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