Supplementary Material to "On the Power Properties of Inference for Parameters with Interval Identified Sets"

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Abstract

This document provides results regarding CI^4_α that were omitted from the main text in Bugni et al. (2024).

1 Proof of Theorems

Proof of part (c) of Theorem 1. We prove (3.3) by contradiction. That is, suppose that $\liminf_{N\to\infty}(P_N(\theta_N \in CI^4_\alpha) - P_N(\theta_N \in CI^j_\alpha)) < 0$ for some $j = 1, 2, 3$. We can then find a subsequence $\{k_N\}_{N\in\mathbb{N}}$ s.t.

$$\lim_{N\to\infty}(P_{k_N}(\theta_{k_N} \in CI^j_\alpha) - P_{k_N}(\theta_{k_N} \in CI^4_\alpha)) \ > \ 0 \quad \text{for some } j = 1, 2, 3. \tag{S.1}$$

The proof is completed by showing that (S.1) cannot hold.

By possibly taking a further subsequence,

$$\begin{pmatrix} \theta_l(P_{k_N}), \theta_u(P_{k_N}), \sigma_l(P_{k_N}), \sigma_u(P_{k_N}), \rho(P_{k_N}), \sqrt{k_N}(\theta_u(P_{k_N}) - \theta_l(P_{k_N})), \\ \sqrt{k_N}(\theta_u(P_{k_N}) - \theta_l(P_{k_N}) - b_{k_N}), \sqrt{k_N}(\theta_l(P_{k_N}) - \theta_{k_N}), \sqrt{k_N}(\theta_{k_N} - \theta_u(P_{k_N})) \end{pmatrix}$$
$$\to (\theta_l, \theta_u, \sigma_l, \sigma_u, \rho, \mu, \eta, \Psi_l, \Psi_u). \tag{S.2}$$

where $\{b_N\}_{N\geq 1}$ is the tuning parameter sequence used to implement CI^3_α.

We then divide the argument into six exhaustive cases, depending on possible values of (μ, Ψ_l, Ψ_u). In this regard, note that $\theta_N \in \Theta_I(P_N)^c$ implies that either (i) $\sqrt{k_N}(\theta_l(P_{k_N}) - \theta_{k_N}) > 0$ or (ii) $\sqrt{k_N}(\theta_{k_N} - \theta_u(P_{k_N})) > 0$. By taking limits, we conclude that either (i) $\Psi_l \geq 0$ or (ii) $\Psi_u \geq 0$. The result is completed by showing that none of the following exhaustive cases satisfy (S.1).

Case 1: $\mu = \infty$, $\Psi_l \geq 0$, and $\eta \in (0, \infty]$. Then, for any $j = 1, 2, 3$, consider the following derivation.

$$\begin{aligned}
P_{k_N}(\theta_{k_N} \in CI^4_\alpha) &\overset{(1)}{\geq} P_{k_N}(\theta_{k_N} \in CI^{4,a}_\alpha) \\
&= P_{k_N}(\hat{\theta}_l - \hat{\sigma}_l c(\hat{\rho})/\sqrt{k_N} \leq \theta_{k_N} \leq \hat{\theta}_u + \hat{\sigma}_u c(\hat{\rho})/\sqrt{k_N}) \\
&= P_{k_N}\left(\left\{ \begin{array}{c} \{\sqrt{k_N}(\hat{\theta}_l - \theta_l(P_{k_N}))/\hat{\sigma}_l - c(\hat{\rho}) \leq -\sqrt{k_N}(\theta_l(P_{k_N}) - \theta_{k_N})/\hat{\sigma}_l\} \cap \\ -\sqrt{k_N}(\theta_l(P_{k_N}) - \theta_{k_N})/\hat{\sigma}_u - \sqrt{k_N}(\theta_u(P_{k_N}) - \theta_l(P_{k_N}))/\hat{\sigma}_u \leq \\ \sqrt{k_N}(\hat{\theta}_u - \theta_u(P_{k_N}))/\hat{\sigma}_u + c(\hat{\rho}) \end{array} \right\} \right) \\
&\overset{(2)}{\to} P(\{z_1 - c(\rho) \leq -\Psi_l/\sigma_l\} \cap \{-(\Psi_l + \mu)/\sigma_u \leq z_2\sqrt{1-\rho^2} + z_1\rho + c(\rho)\}) \\
&\overset{(3)}{=} P(\{z_1 - c(\rho) \leq -\Psi_l/\sigma_l\}) \\
&= \Phi(c(\rho) - \Psi_l/\sigma_l) \\
&\overset{(4)}{\geq} \Phi(\Phi^{-1}(1-\alpha) - \Psi_l/\sigma_l) \\
&\overset{(5)}{=} \lim P_{k_N}(\theta_{k_N} \in CI^j_\alpha),
\end{aligned} \tag{S.3}$$

where (1) holds by $CI^{4,a}_\alpha \subseteq CI^4_\alpha$ with $CI^{4,a}_\alpha = [\hat{\theta}_l - \hat{\sigma}_l c(\hat{\rho})/\sqrt{k_N}, \hat{\theta}_u + \hat{\sigma}_u c(\hat{\rho})/\sqrt{k_N}]$ with $c(\cdot)$ as in Definition S.1, (2) by (S.2), Lemma S.1, and OBS (Definition 1), (3) by $\mu = \infty$, (4) by Lemma S.2, and (5) by part (a) of Lemma 1, $F_1(\infty, \sigma_l, \sigma_u) = \Phi^{-1}(1-\alpha)$, part (a) of Lemma 2, and part (c) of Lemma 3. Note that (S.3) implies that (S.1) fails.

Case 2: $\mu = \infty$, $\Psi_l \geq 0$, and $\eta \in [-\infty, 0]$. By this and Lemma 4, we deduce that $\rho = 1$ and $\sigma = \sigma_l = \sigma_u$. Then, for any $j = 1, 2, 3$, consider the following derivation.

$$P_{k_N}(\theta_{k_N} \in CI^4_\alpha) \overset{(1)}{\geq} P_{k_N}(\theta_{k_N} \in CI^{4,a}_\alpha)$$

$$
\begin{aligned}
&= P_{k_N}\left(\begin{array}{c}\{\sqrt{k_N}(\hat{\theta}_l-\theta_l(P_{k_N}))/\hat{\sigma}_l-c(\hat{\rho})\leq-\sqrt{k_N}(\theta_l(P_{k_N})-\theta_{k_N})/\hat{\sigma}_l\}\cap\\ \left\{\begin{array}{c}-\sqrt{k_N}(\theta_l(P_{k_N})-\theta_{k_N})/\hat{\sigma}_u-\sqrt{k_N}(\theta_u(P_{k_N})-\theta_l(P_{k_N}))/\hat{\sigma}_u\\ \leq\sqrt{k_N}(\hat{\theta}_u-\theta_u(P_{k_N}))/\hat{\sigma}_u+c(\hat{\rho})\end{array}\right\}\end{array}\right)\\
&\overset{(2)}{\to} P(\{z_1-c(1)\leq-\Psi_l/\sigma\})\\
&\overset{(3)}{=} P(\{z_1-\Phi^{-1}(1-\alpha/2)\leq-\Psi_l/\sigma\})\\
&= \Phi(\Phi^{-1}(1-\alpha/2)-\Psi_l/\sigma)\\
&\overset{(4)}{\geq} \limsup_{N\to\infty} P_{k_N}(\theta_{k_N}\in CI_\alpha^j),
\end{aligned}
\tag{S.4}
$$

where (1) holds by $CI_\alpha^{4,a}\subseteq CI_\alpha^4$ with $CI_\alpha^{4,a}=[\hat{\theta}_l-\hat{\sigma}_l c(\hat{\rho})/\sqrt{k_N},\hat{\theta}_u+\hat{\sigma}_u c(\hat{\rho})/\sqrt{k_N}]$ with $c(\cdot)$ as in Definition S.1, (2) by (S.2), Lemma S.1, and OBS (Definition 1), (3) by Lemma S.3, and (4) by $\Phi^{-1}(1-\alpha/2)\geq\Phi^{-1}(1-\alpha)$, Lemma 1, $F_1(\infty,\sigma,\sigma)=\Phi^{-1}(1-\alpha)$, part (a) of Lemmas 1 and 2, and parts (b) and (d) of Lemma 3. Note that (S.4) implies that (S.1) fails.

Case 3: $\mu<\infty$ and $\Psi_l\geq 0$. By $\mu\in\mathbb{R}_+$, it follows that $\theta_u(P_{k_N})-\theta_l(P_{k_N})\to 0$. By this and Lemma 4, we deduce that $\rho=1$ and $\sigma=\sigma_l=\sigma_u$. Also, by repeating the argument in (A-5), we deduce that $\Phi^{-1}(1-\alpha/2)\geq G(\mu/\sigma)$. Then, for any $j=1,2,3$, consider the following derivation.

$$
\begin{aligned}
P_{k_N}(\theta_{k_N}\in CI_\alpha^4) &\overset{(1)}{\geq} P_{k_N}(\theta_{k_N}\in CI_\alpha^{4,a})\\
&= P_{k_N}\left(\begin{array}{c}\{\sqrt{k_N}(\hat{\theta}_l-\theta_l(P_{k_N}))/\hat{\sigma}_l-c(\hat{\rho})\leq-\sqrt{k_N}(\theta_l(P_{k_N})-\theta_{k_N})/\hat{\sigma}_l\}\cap\\ \left\{\begin{array}{c}-\sqrt{k_N}(\theta_l(P_{k_N})-\theta_{k_N})/\hat{\sigma}_u-\sqrt{k_N}(\theta_u(P_{k_N})-\theta_l(P_{k_N}))/\hat{\sigma}_u\\ \leq\sqrt{k_N}(\hat{\theta}_u-\theta_u(P_{k_N}))/\hat{\sigma}_u+c(\hat{\rho})\end{array}\right\}\end{array}\right)\\
&\overset{(2)}{\to} P(\{z_1-c(1)\leq-\Psi_l/\sigma\}\cap\{-(\Psi_l+\mu)/\sigma\leq z_1+c(1)\})\\
&\overset{(3)}{=} \Phi((\Psi_l+\mu)/\sigma+\Phi^{-1}(1-\alpha/2))-\Phi(\Psi_l/\sigma-\Phi^{-1}(1-\alpha/2))\\
&\overset{(4)}{\geq} \lim_{N\to\infty} P_{k_N}(\theta_{k_N}\in CI_\alpha^j),
\end{aligned}
\tag{S.5}
$$

where (1) holds by $CI_\alpha^{4,a}\subseteq CI_\alpha^4$ with $CI_\alpha^{4,a}=[\hat{\theta}_l-\hat{\sigma}_l c(\hat{\rho})/\sqrt{k_N},\hat{\theta}_u+\hat{\sigma}_u c(\hat{\rho})/\sqrt{k_N}]$ with $c(\cdot)$ as in Definition S.1, (2) by (S.2), Lemma S.1, and OBS (Definition 1), (3) by Lemma S.3, and (4) by $\Phi^{-1}(1-\alpha/2)\geq G(\mu/\sigma)$, Lemma 1, $F_1(\mu,\sigma,\sigma)=G(\mu/\sigma)$, part (a) of Lemma 2, and part (e) of Lemma 3. Note that (S.5) implies that (S.1) fails.

Case 4: $\mu=\infty$, $\Psi_u\geq 0$, and $\eta\in(0,\infty]$. This case is analogous to case 1 and, therefore, omitted.

Case 5: $\mu = \infty$, $\Psi_u \geq 0$, and $\eta \in [-\infty, 0]$. This case is analogous to case 2 and, therefore, omitted.

Case 6: $\mu < \infty$ and $\Psi_u \geq 0$. This case is analogous to case 3 and, therefore, omitted.

To conclude the proof, it suffices to verify (3.3) holds strictly for suitably chosen sequences $\{(P_N, \theta_N) \in \mathcal{P} \times \Theta_I(P_N)^c\}_{N \in \mathbb{N}}$. We use Example 1 with $\underline{Y} = 0$, $\overline{Y} = 1$, $\{Y_i | Z_i = 1\} \sim Be(1/2)$, and $Z_i \sim Be(1 - \pi(P_N))$, where $\pi(P_N) = (\ln N)/\sqrt{N} + a_1/\sqrt{N} \downarrow 0$ for some $a_1 > 0$. We consider coverage of $\theta_N = \theta_l(P_N) - a_2/\sqrt{N} \in \Theta_I(P_N)^c$ for some $a_2 > 0$ and CI_α^3 implemented with $b_N = (\ln N)/\sqrt{N}$ and the full sample the data (i.e., $\lambda = 1$).

By repeating arguments presented in the proof of Theorem 2, we deduce that the vector $(\theta_l, \theta_u, \sigma_l, \sigma_u, \rho, \mu, \eta, \Psi_l) = (1/2, 1/2, 1/2, 1/2, 1, \infty, a_1, a_2)$, and part (c) of Lemma 3 implies that

$$\lim_{N \to \infty} P_N(\theta_N \in CI_\alpha^3) \;=\; \Phi(\Phi^{-1}(1-\alpha) - 2\sqrt{\lambda} a_2). \tag{S.6}$$

In turn, Lemmas 1, 2, S.3, and S.4 imply that

$$\begin{aligned} \lim_{N \to \infty} P_N(\theta_N \in CI_\alpha^j) &= \Phi(\Phi^{-1}(1-\alpha) - 2\sqrt{\lambda} a_2) \qquad \text{for } j = 1, 2, \\ \lim_{N \to \infty} P_N(\theta_N \in CI_\alpha^4) &= \Phi(\Phi^{-1}(1-\alpha/2) - 2\sqrt{\lambda} a_2). \end{aligned} \tag{S.7}$$

The desired result follows from (S.6), (S.7), and $\Phi^{-1}(1-\alpha) < \Phi^{-1}(1-\alpha/2)$. For instance, using $a_1 = 1.5$, $a_2 = 0.01$, and $\alpha = 0.05$ yields $\lim_{N\to\infty} P_N(\theta_N \in CI_\alpha^4) = 0.974 > 0.948 = \lim_{N\to\infty} P_N(\theta_N \in CI_\alpha^1) = \lim_{N\to\infty} P_N(\theta_N \in CI_\alpha^2) = \lim_{N\to\infty} P_N(\theta_N \in CI_\alpha^3)$. ■

Proof of Theorem 5. To show this result, we construct specific sequences where (3.9) and (3.10) can occur. We focus on sequences $\{(P_N, \theta_N) \in \mathcal{P} \times \Theta_I(P_N)^c\}_{N \in \mathbb{N}}$ s.t.

$$\begin{aligned} &\left(\begin{array}{c} \theta_l(P_N), \theta_u(P_N), \sigma_l^E(P_N), \sigma_u^E(P_N), \rho^E(P_N), \sigma_l^I(P_N), \sigma_u^I(P_N), \rho^I(P_N) \\ \sqrt{N}(\theta_u(P_N) - \theta_l(P_N)), \sqrt{N}(\theta_l(P_N) - \theta_N), \sqrt{N}(\theta_N - \theta_u(P_N)) \end{array} \right) \\ &\to (\theta_l, \theta_u, \sigma_l^E, \sigma_u^E, \rho^E, \sigma_l^I, \sigma_u^I, \rho^I, \mu, \Psi_l, \Psi_u). \end{aligned} \tag{S.8}$$

with $\Psi_l \geq 0$ and $\mu \in \mathbb{R}_+$. By Lemma 4, $\rho^E = \rho^I = 1$ and $\sigma_l^E = \sigma_u^E$, and $\sigma_l^I = \sigma_u^I$.

We can construct concrete sequences in the context of Example 1. In particular, we use Example 1 with $\underline{Y} = 0$, $\overline{Y} = 1$, $\{Y_i | Z_i = 1\} \sim Be(1/2)$, and $Z_i \sim Be(1 - \pi(P_N))$, where $\pi(P_N) = a_1/\sqrt{N} \downarrow 0$ for some $a_1 > 0$. We consider coverage of $\theta_N = \theta_l(P_N) - a_2/\sqrt{N} \in \Theta_I(P_N)^c$ for some $a_2 > 0$ and CI_α^4 implemented with the subsample of the data $\{(Y_i, Z_i)\}_{i=1}^{\lfloor \lambda N \rfloor}$ for $\lambda \in (0, 1]$. In this context, we consider two estimators for the bounds. The efficient estimator uses the full sample, while the inefficient estimator uses only a fraction $\lambda = a_3 \in (0,1)$ of the sample. By repeating arguments in Theorem 2, we deduce that (S.8) holds with

$$(\theta_l, \theta_u, \sigma_l^E, \sigma_u^E, \rho^E, \sigma_l^I, \sigma_u^I, \rho^I, \mu, \Psi_l, \Psi_u) \;=\; \Big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2\sqrt{a_3}}, \frac{1}{2\sqrt{a_3}}, 1, a_1, a_2, -(a_1{+}a_2)\Big).$$

Part (b) of Lemma S.4 then yields

$$\begin{aligned}
\lim_{N\to\infty} P_N(\theta_N \in CI_\alpha^{4,E}) &\;=\; \Phi\big(2(a_2+a_1) + \Phi^{-1}(1-\alpha/2)\big) - \Phi\big(2a_2 - \Phi^{-1}(1-\alpha/2)\big),\\
\lim_{N\to\infty} P_N(\theta_N \in CI_\alpha^{4,I}) &\;=\; \Phi\big(2\sqrt{a_3}(a_2+a_1) + \Phi^{-1}(1-\alpha/2)\big) - \Phi\big(2\sqrt{a_3}a_2 - \Phi^{-1}(1-\alpha/2)\big).
\end{aligned}$$

We now verify the strict inequalities. To obtain (3.9), set $a_1 = a_2 = 1$, $a_3 = 0.3$, and $\alpha = 0.05$, which give

$$\lim_{N\to\infty} P_N(\theta_N \in CI_\alpha^{4,E}) \;=\; 0.48 \;<\; 0.81 \;=\; \lim_{N\to\infty} P_N(\theta_N \in CI_\alpha^{4,I}).$$

To obtain (3.10), set $a_1 = 0.15$, $a_2 = 0.01$, $a_3 = 0.3$, and $\alpha = 0.05$, which give

$$\lim_{N\to\infty} P_N(\theta_N \in CI_\alpha^{4,E}) \;=\; 0.963 \;>\; 0.958 \;=\; \lim_{N\to\infty} P_N(\theta_N \in CI_\alpha^{4,I}).$$

This completes the proof. ■

2 Auxiliary material

This section collects auxiliary definitions and intermediate results that are useful for proving parts (e)-(f) of Theorem 1 and Theorem 5.

Definition S.1. *For any $\rho \in [-1,1]$, $c(\rho)$ is the unique c that solves*

$$\inf_{\Delta\ge 0} P\left(\begin{array}{c}\{z_1 - \Delta - c \le 0 \le \rho z_1 + z_2\sqrt{1-\rho^2} + c\} \cup \\ \{|(1+\rho)z_1 + z_2\sqrt{1-\rho^2} - \Delta| \le \sqrt{2+2\rho}\,\Phi^{-1}(1-\alpha/2)\}\end{array}\right) \;=\; 1-\alpha,$$

where $z = (z_1, z_2) \sim \mathcal{N}(\mathbf{0}_{2\times1}, \mathbf{I}_{2\times2})$.

Remark S-1. *By definition, Stoye (2020) uses $c^4 = c(\hat\rho)$. To see this, observe that $z = (z_1, z_2) \sim \mathcal{N}(\mathbf{0}_{2\times1}, \mathbf{I}_{2\times2})$ is equivalent to $\tilde z(\rho) = (\tilde z_1(\rho), \tilde z_2(\rho)) := (z_1, \rho z_1 + z_2\sqrt{1-\rho^2}) \sim$*

$\mathcal{N}(\mathbf{0}_{2\times 1},[1,\rho;\rho,1])$. Therefore, $c(\rho)$ can equivalently be characterized as the unique c solving

$$\inf_{\Delta\geq 0} P\left(\begin{array}{c}\{\tilde{z}_1(\rho)-\Delta-c\leq 0\leq \tilde{z}_2(\rho)+c\}\cup\\ \{|\tilde{z}_1(\rho)+\tilde{z}_2(\rho)-\Delta|\leq\sqrt{2+2\rho}\,\Phi^{-1}(1-\alpha/2)\}\end{array}\right) \;=\; 1-\alpha.$$

If we set $\rho=\hat{\rho}$ and condition on $\hat{\rho}$, the above expression coincides with the one in Stoye (2020).

Lemma S.1. *$c(\rho)$ in Definition S.1 is a continuous function of $\rho\in[-1,1]$.*

Proof. Recall $\bar{\mathbb{R}}_+=[0,\infty]$. Define the function $G:[-1,1]\times\bar{\mathbb{R}}_+\times\bar{\mathbb{R}}_+\to[0,1]$ as follows:

$$G(\rho,\Delta,c) \;=\; P\left(\begin{array}{c}\{z_1-\Delta-c\leq 0\leq \rho z_1+z_2\sqrt{1-\rho^2}+c\}\cup\\ \{|(1+\rho)z_1+z_2\sqrt{1-\rho^2}-\Delta|\leq\sqrt{2+2\rho}\Phi^{-1}(1-\alpha/2)\}\end{array}\right),$$

where $G(\rho,\Delta,\infty)=1$ and $G(\rho,\infty,c)=\Phi(c)$. Define also $\widetilde{G}(\rho,c)=\inf_{\Delta\geq 0}G(\rho,\Delta,c)$. It is easy to see that $\widetilde{G}$ is continuous in all its arguments.[1] By definition,

$$c(\rho) \;=\; \Big\{c:\widetilde{G}(\rho,c)=1-\alpha\Big\},$$

where the uniqueness of the solution follows from Stoye (2020).

Suppose $c(\rho)$ is not a continuous function of $\rho\in[-1,1]$, i.e., $\{\rho_m\}_{m\in\mathbb{N}}$ with $\rho_m\to\rho\in[-1,1]$ and $c(\rho_m)\not\to c(\rho)$. By possibly taking a subsequence, we have that $c(\rho_m)\to\bar{c}\neq c(\rho)$ as $m\to\infty$, where $\bar{c}\in[0,\infty]$. By definition of $c(\rho_m)$,

$$\widetilde{G}(\rho_m,c(\rho_m)) \;=\; 1-\alpha. \tag{S.9}$$

Since $c(\rho)$ is unique and $\bar{c}\neq c(\rho)$, we have $\widetilde{G}(\rho,\bar{c})\neq 1-\alpha$. There are two cases: $\widetilde{G}(\rho,\bar{c})>1-\alpha$ or $\widetilde{G}(\rho,\bar{c})<1-\alpha$. To complete the proof, it suffices to show that both cases are contradictory.

Case 1: $\widetilde{G}(\rho,\bar{c})>1-\alpha$. Then, $\exists\varepsilon>0$ s.t. $\widetilde{G}(\rho,\bar{c})\geq 1-\alpha+\varepsilon$. By continuity of $\widetilde{G}$, $\lim_{m\to\infty}\widetilde{G}(\rho_m,c(\rho_m))=\widetilde{G}(\rho,\bar{c})\geq 1-\alpha+\varepsilon$. Therefore, $\exists M$ s.t. for all $m\geq M$, $\widetilde{G}(\rho_m,c(\rho_m))\geq 1-\alpha+\varepsilon/2$, which is a contradiction to (S.9).

[1] For any sequence $(\rho_n,c_n)\to(\rho,c)$, $\exists\Delta_n,\Delta_0\in\bar{\mathbb{R}}_+$ s.t. $\widetilde{G}(\rho_n,c_n)=G(\rho_n,\Delta_n,c_n)$ and $\widetilde{G}(\rho,c)=G(\rho,\Delta_0,c)$. By taking subsequence if necessary we can assume that $\Delta_n\to\bar{\Delta}$ for some $\bar{\Delta}$. When $\rho>-1$, the continuity of G on $(-1,1]\times\bar{\mathbb{R}}_+\times\bar{\mathbb{R}}_+$ and definition of $\widetilde{G}$ imply $\widetilde{G}(\rho,c)\leq G(\rho,\bar{\Delta},c)=\lim_{n\to\infty}\widetilde{G}(\rho_n,c_n)\leq\lim_{n\to\infty}G(\rho_n,\Delta_0,c_n)=\widetilde{G}(\rho,c)$; hence $\widetilde{G}(\rho,c)=\lim_{n\to\infty}\widetilde{G}(\rho_n,c_n)$. When $\rho=-1$, the result follows by similar arguments, using $\widetilde{G}(-1,c)=\Phi(c)$, the continuity of G on $[-1,1]\times(0,\infty]\times\bar{\mathbb{R}}_+$ for the upper bound, and the fact that, uniformly over $\Delta\geq 0$, the first event contains $z_1\leq c,\ \rho z_1+z_2\sqrt{1-\rho^2}\geq -c$ for the lower bound.

Case 2: $\widetilde{G}(\rho,\bar{c}) < 1-\alpha$. Then, $\exists\varepsilon > 0$ s.t. $\widetilde{G}(\rho,\bar{c}) \leq 1-\alpha-\varepsilon$. By continuity of $\widetilde{G}$, $\lim_{m\to\infty}\widetilde{G}(\rho_m, c(\rho_m)) = \widetilde{G}(\rho,\bar{c}) \leq 1-\alpha-\varepsilon/2$. Therefore, $\exists M$ s.t. for all $m \geq M$, $\widetilde{G}(\rho_m, c(\rho_m)) \leq 1-\alpha-\varepsilon/4$, which is a contradiction to (S.9). ■

Lemma S.2. *For any $\rho \in [-1,1]$ and $\alpha \in (0,1)$, $c(\rho)$ in Definition S.1 satisfies $c(\rho) \geq \Phi^{-1}(1-\alpha)$.*

Proof. Fix $\rho \in [-1,1]$ arbitrarily. Assume the result is false, i.e., $c(\rho) < \Phi^{-1}(1-\alpha)$. Then, $\exists\varepsilon > 0$ s.t. $c(\rho) \leq \Phi^{-1}(1-\alpha)-\varepsilon$. Define

$$\Pi(c) \;=\; \inf_{\Delta\geq 0} P\left(\begin{array}{c} \{z_1-\Delta-c \leq 0 \leq \rho z_1 + z_2\sqrt{1-\rho^2}+c\}\cup \\ \left\{\begin{array}{c} \Delta-\sqrt{2+2\rho}\Phi^{-1}(1-\alpha/2) \leq \\ z_1(1+\rho)+z_2\sqrt{1-\rho^2} \leq \Delta+\sqrt{2+2\rho}\Phi^{-1}(1-\alpha/2) \end{array}\right\} \end{array}\right).$$

It is easy to see that $\Pi(c)$ is nondecreasing in c.

By $\Pi(c(\rho)) = 1-\alpha$ and $c(\rho) \leq \Phi^{-1}(1-\alpha)-\varepsilon$, the monotonicity of $\Pi(c)$ implies that $\Pi(\Phi^{-1}(1-\alpha)-\varepsilon) \geq 1-\alpha$. Then, for any $\Delta \geq 0$,

$$\Pi(\Phi^{-1}(1-\alpha)-\varepsilon)$$

is less than or equal to

$$P\left(\begin{array}{c} \{z_1-\Delta-\Phi^{-1}(1-\alpha)+\varepsilon \leq 0 \leq \rho z_1+z_2\sqrt{1-\rho^2}+\Phi^{-1}(1-\alpha)-\varepsilon\}\cup \\ \{\Delta-\sqrt{2+2\rho}\Phi^{-1}(1-\alpha/2) \leq z_1(1+\rho)+z_2\sqrt{1-\rho^2} \leq \Delta+\sqrt{2+2\rho}\Phi^{-1}(1-\alpha/2)\} \end{array}\right).$$

Now evaluate this inequality for a sequence $\Delta_m \to \infty$ to obtain

$$\Pi(\Phi^{-1}(1-\alpha)-\varepsilon) \leq P(0 \leq \rho z_1+z_2\sqrt{1-\rho^2}+\Phi^{-1}(1-\alpha)-\varepsilon) = \Phi\left(\Phi^{-1}(1-\alpha)-\varepsilon\right)\ ,$$

where we used $\tilde{z} = \rho z_1 + z_2\sqrt{1-\rho^2} \sim N(0,1)$ in the last equality. Finally, we obtain

$$\Phi\left(\Phi^{-1}(1-\alpha)\right) \;\leq\; \Pi(\Phi^{-1}(1-\alpha)-\varepsilon) \;\leq\; \Phi\left(\Phi^{-1}(1-\alpha)-\varepsilon\right),$$

which is a contradiction. ■

Lemma S.3. *For any $\alpha \in (0,1)$, $c(\rho)$ in Definition S.1 satisfies $c(1) = \Phi^{-1}(1-\alpha/2)$.*

Proof. By definition,

$$c(1) \;=\; \left\{c : \inf_{\Delta\geq 0} P\left(\begin{array}{c} \{-c \leq z \leq \Delta+c\} \quad\cup \\ \{\Delta/2-\Phi^{-1}(1-\alpha/2) \leq z \leq \Phi^{-1}(1-\alpha/2)+\Delta/2\} \end{array}\right) = 1-\alpha\right\}.$$

Given any $c \in \mathbb{R}$, consider the minimization of

$$G(\Delta) = \Phi(\max\{\Phi^{-1}(1-\alpha/2) + \Delta/2, \Delta + c\}) - \Phi(\min\{\Delta/2 - \Phi^{-1}(1-\alpha/2), -c\})$$

with respect to $\Delta \geq 0$. We have two cases.

Case 1: $2\Phi^{-1}(1-\alpha/2) - 2c \geq 0$. If $\Delta \geq 2\Phi^{-1}(1-\alpha/2) - 2c$, then, $\Delta + c \geq \Phi^{-1}(1-\alpha/2) + \Delta/2$ and $-c \leq \Delta/2 - \Phi^{-1}(1-\alpha/2)$, and so $G(\Delta) = \Phi(\Delta + c) - \Phi(-c)$. Hence G is increasing in Δ on $[2\Phi^{-1}(1-\alpha/2) - 2c, \infty)$. If $\Delta < 2\Phi^{-1}(1-\alpha/2) - 2c$, then, $G(\Delta) = \Phi(\Phi^{-1}(1-\alpha/2) + \Delta/2) - \Phi(\Delta/2 - \Phi^{-1}(1-\alpha/2))$. The function G is decreasing in Δ on $[0, 2\Phi^{-1}(1-\alpha/2) - 2c]$. Therefore, in this case, G is minimized by setting $\Delta = 2\Phi^{-1}(1-\alpha/2) - 2c$.

Case 2: $2\Phi^{-1}(1-\alpha/2) - 2c < 0$. Since $\Delta \geq 0$, we have $\Delta \geq 2\Phi^{-1}(1-\alpha/2) - 2c$. The same argument with the constraint $\Delta \geq 0$ yields $\Delta = 0$.

Combining all cases,

$$\inf_{\Delta \geq 0} G(\Delta) = \left\{ \begin{array}{c} I\{\Phi^{-1}(1-\alpha/2) \geq c\}(\Phi(2\Phi^{-1}(1-\alpha/2) - c) - \Phi(-c)) \\ +I\{\Phi^{-1}(1-\alpha/2) < c\}(\Phi(c) - \Phi(-c)) \end{array} \right\}. \quad \text{(S.10)}$$

Now, we solve for c s.t. (S.10) equals $1-\alpha$ which, by definition, equals $c(1)$.

We now show that $\Phi^{-1}(1-\alpha/2) \geq c$. Suppose otherwise that $\Phi^{-1}(1-\alpha/2) < c$. Then, the right-hand side of (S.10) equals $\Phi(c) - \Phi(-c)$. Then, the desired solution c satisfies $\Phi(c) - \Phi(-c) = 1-\alpha$, which yields $\Phi(c) = 1-\alpha/2$, and, so, $c = \Phi^{-1}(1-\alpha/2)$, which contradicts the premise that $\Phi^{-1}(1-\alpha/2) < c$.

Since $\Phi^{-1}(1-\alpha/2) \geq c$, the right-hand side of (S.10) equals $\Phi(2\Phi^{-1}(1-\alpha/2) - c) - \Phi(-c)$. Then, the desired solution c satisfies

$$\Phi(2\Phi^{-1}(1-\alpha/2) - c) - \Phi(-c) = 1-\alpha. \quad \text{(S.11)}$$

Note that the left-hand side of (S.11) is strictly increasing in c whenever $\Phi^{-1}(1-\alpha/2) \geq c$ holds. Therefore, the unique solution is $c = \Phi^{-1}(1-\alpha/2)$, as desired. ■

Lemma S.4. *Let $\alpha \in (0, 0.5)$ and Assumption 1 hold. Let $\{(P_N, \theta_N) \in \mathcal{P} \times \Theta_I(P_N)^c\}_{N \in \mathbb{N}}$ be a sequence s.t.*

$$\begin{aligned} &\left(\begin{array}{c} \theta_l(P_N), \theta_u(P_N), \sigma_l(P_N), \sigma_u(P_N), \rho(P_N), \\ \sqrt{N}(\theta_u(P_N) - \theta_l(P_N)), \sqrt{N}(\theta_l(P_N) - \theta_N), \sqrt{N}(\theta_N - \theta_u(P_N)) \end{array} \right) \\ &\to (\theta_l, \theta_u, \sigma_l, \sigma_u, \rho, \mu, \Psi_l, \Psi_u). \end{aligned} \quad \text{(S.12)}$$

Moreover, assume that $\Psi_l \geq 0$. *Then,*

(a) If $\mu = \infty$, $P_N(\theta_N \in CI_\alpha^4) \to \Phi(c(\rho) - \Psi_l/\sigma_l)$.

(b) If $\mu \in \mathbb{R}_+$, *then* $\rho = 1$, $\sigma_l = \sigma_u$, *and, if we denote* $\sigma = \sigma_l = \sigma_u$, *we get*

$$P_N(\theta_N \in CI_\alpha^4) \;\to\; \Phi((\Psi_l + \mu)/\sigma + \Phi^{-1}(1-\alpha/2)) - \Phi(\Psi_l/\sigma - \Phi^{-1}(1-\alpha/2)).$$

Proof. As a preliminary result, note that

$$\Psi_u \;=\; -\lim(\sqrt{N}(\theta_u(P_N) - \theta_l(P_N)) + \sqrt{N}(\theta_l(P_N) - \theta_N)) \;=\; -\mu - \Psi_l. \tag{S.13}$$

We divide the proof into two parts.

Part (a): In this case, $\mu = \infty$. Then $\Psi_l \geq 0$ and, by (S.13), $\Psi_u = -\infty$. Hence,

$$\begin{aligned}
&P_N(\theta_N \in CI_\alpha^4)\\
&= P_N(\theta_N \in CI_\alpha^{4,a} \cup CI_\alpha^{4,b})\\
&= P_N\left(\begin{array}{c}
\left\{\begin{array}{c}
\{\sqrt{N}(\hat{\theta}_l - \theta_l(P_N))/\hat{\sigma}_l - c(\hat{\rho}) \leq -\sqrt{N}(\theta_l(P_N) - \theta_N)/\hat{\sigma}_l\}\cap \\
\left\{\begin{array}{c}
-\sqrt{N}(\theta_l(P_N) - \theta_N)/\hat{\sigma}_u - \sqrt{N}(\theta_u(P_N) - \theta_l(P_N))/\hat{\sigma}_u \\
\leq \sqrt{N}(\hat{\theta}_u - \theta_u(P_N))/\hat{\sigma}_u + c(\hat{\rho})
\end{array}\right\}
\end{array}\right\} \\
\cup \left\{\begin{array}{c}
\sqrt{N}(\theta_N - \theta_u(P_N))/\hat{\sigma}_u - \sqrt{N}(\theta_l(P_N) - \theta_N)/\hat{\sigma}_l - \sqrt{2+2\hat{\rho}}\Phi^{-1}(1-\alpha/2) \\
\leq \sqrt{N}(\hat{\theta}_l - \theta_l(P_N))/\hat{\sigma}_l + \sqrt{N}(\hat{\theta}_u - \theta_u(P_N))/\hat{\sigma}_u \leq \\
\sqrt{N}(\theta_N - \theta_u(P_N))/\hat{\sigma}_u - \sqrt{N}(\theta_l(P_N) - \theta_N)/\hat{\sigma}_l + \sqrt{2+2\hat{\rho}}\Phi^{-1}(1-\alpha/2)
\end{array}\right\}
\end{array}\right)\\
&\overset{(1)}{\to} P\left(\begin{array}{c}
\{\{z_1 - c(\rho) \leq -\Psi_l/\sigma_l\} \cap \{-\Psi_l/\sigma_u - \mu/\sigma_u \leq \rho z_1 + z_2\sqrt{1-\rho^2} + c(\rho)\}\}\cup \\
\left\{\begin{array}{c}
\Psi_u/\sigma_u - \Psi_l/\sigma_l - \sqrt{2+2\rho}\Phi^{-1}(1-\alpha/2) \leq (1+\rho)z_1 + z_2\sqrt{1-\rho^2} \\
\leq \Psi_u/\sigma_u - \Psi_l/\sigma_l + \sqrt{2+2\rho}\Phi^{-1}(1-\alpha/2)
\end{array}\right\}
\end{array}\right)\\
&\overset{(2)}{=} P(z_1 - c(\rho) \leq -\Psi_l/\sigma_l) \;=\; \Phi(c(\rho) - \Psi_l/\sigma_l),
\end{aligned}$$

as desired, where (1) holds by (S.12), Lemma S.1, and OBS (Definition 1), and (2) by $\mu = \infty$, $\Psi_l \geq 0$, and (S.13), which implies that $\Psi_u = -\infty$.

Part (b): In this case, $\mu \in \mathbb{R}_+$. Lemma 4 then implies that $\rho = 1$ and $\sigma_l = \sigma_u$. Let $\sigma = \sigma_l = \sigma_u$. By a similar derivation as in part (a),

$$\begin{aligned}
P_N(\theta_N \in CI_\alpha^4) &\overset{(1)}{\to} P\left(\begin{array}{c}
\{\{z_1 - c(1) \leq -\Psi_l/\sigma\} \cap \{-(\Psi_l + \mu)/\sigma \leq z_1 + c(1)\}\} \\
\cup \left\{\begin{array}{c}
(\Psi_u - \Psi_l)/\sigma - 2\Phi^{-1}(1-\alpha/2) \leq 2z_1 \\
\leq (\Psi_u - \Psi_l)/\sigma + 2\Phi^{-1}(1-\alpha/2)
\end{array}\right\}
\end{array}\right)\\
&\overset{(2)}{=} P(-(\Psi_l + \mu)/\sigma - \Phi^{-1}(1-\alpha/2) \leq z_1 \leq -\Psi_l/\sigma + \Phi^{-1}(1-\alpha/2))
\end{aligned}$$

$$= \Phi((\Psi_l + \mu)/\sigma + \Phi^{-1}(1-\alpha/2)) - \Phi(\Psi_l/\sigma - \Phi^{-1}(1-\alpha/2)),$$

where (1) holds by (S.12), Lemma S.1, and OBS (Definition 1), and (2) by Lemma S.3, $\mu \geq 0$, $\Psi_l \geq 0$, and (S.13), which implies that $\Psi_u = -\mu - \Psi_l$. ■

References

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